Generating Dirichlet Characters

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A Dirichlet character χ is an arithmetic function from the integers to the complex numbers. The concept was introduced by P.G.L. Dirichlet in his proof of *Dirichlet's theorem* that there are an infinite number of primes in arithmetical progressions with first term h and common difference k where $(h, k) = 1$.

In this paper we show a technique to compute the values of χ for any modulus k.

By definition, a Dirichlet character modulo k is a completely multiplicative function, periodic with period k, which is not identically zero and which equals zero if its argument is not coprime with k. That is, $\chi(m)\chi(n) = \chi(mn)$ and $\chi(n) = \chi(n+k)$ for all integers m, n; $\chi \not\equiv 0$, and $\chi(n) = 0$ if $(n, k) > 0$.

From this definition, it follows that $\chi(1) = 1$ and that each $\chi(n)$ is a $\phi(k)$th root of unity when $(n, k) = 1$, where ϕ is Euler's totient function, the number of positive integers less than or equal to k that are coprime to k. There are exactly $\phi(k)$ characters for a given modulus k,

The *principal character*, denoted χ_1, has value one if n and k are coprime and value zero otherwise. Characters other than the principal character are known as *nonprincipal* characters and are denoted $\chi_2, \chi_3, \ldots, \chi_{\phi(k)}$. This latter numbering is arbitrary.

The nonprincipal characters may have complex values. There is, however, always at least one real-valued nonprincipal character. Interestingly, if k is a prime, one of these will correspond to the Legendre symbol so that $\chi(n) = (n|k)$. This is known as the *quadratic character*.

The $\phi(k)$ characters modulo k form an Abelian group isomorphic to the multiplicative group $(\mathbb{Z}/k\mathbb{Z})^*$ of the residues coprime to k, with the principal character χ_1 as the identity. The inverse of $\chi(a)$ is its complex conjugate $\overline{\chi}(a)$ so that $\overline{\chi}(a)\chi(a) = \chi(a)\overline{\chi}(a) = \chi(1)$.

The above properties give us a procedure to define all the values of χ for a given k. It is useful to consider the tuple $\mathbf{X}(a)$ of $\phi(k)$ values where

$$\mathbf{X}(a) = \left[\chi_1(a), \chi_2(a), \ldots, \chi_{\phi(k)}(a)\right].$$

The product of two tuples is then simply the tuple consisting of the product of each element, i.e. $\mathbf{X}(a)\mathbf{X}(b) = [\chi_1(a)\chi_1(b), \chi_2(a)\chi_2(b), \ldots]$. There are always exactly $\phi(k)$ unique tuples. The tuples also behave as a group isomorphic to $(\mathbb{Z}/k\mathbb{Z})^*$.

In the same way that the members of an Abelian group can be completely defined by the powers of one or more generators and their products, so can all the nonzero Dirichlet characters.

Consider the group $G = (\mathbb{Z}/k\mathbb{Z})^*$. If G is cyclic with generator a, then we set the elements of $\mathbf{X}(a)$ equal to each of the $\phi(k)$th roots of unity, with $\chi_1(a) = 1$. The other nonzero values are now completely defined by $\mathbf{X}(a^2) = \mathbf{X}(a)^2$, $\mathbf{X}(a^3) = \mathbf{X}(a)^3$, and so forth, with the bonus of a check on our calculations that $\mathbf{X}(a^{\phi(k)})$ should equal $\mathbf{X}(1) = [1, 1, \ldots, 1]$. Another check is that all the tuples should be unique.

For example, for $k = 10$ we have $\phi(k) = 4$ and the group $G = \{1, 3, 7, 9\}$ is cyclic with generator 3. So we set $\mathbf{X}(3) = [1, i, -1, -i]$, the fourth roots of unity. The order of the elements is immaterial except the first must be one. Now we can derive $\mathbf{X}(9) = \mathbf{X}(3)^2 = [1, -1, 1, -1]$, and $\mathbf{X}(7) = \mathbf{X}(3)^3 = \mathbf{X}(9)\mathbf{X}(3) = [1, -i, -1, i]$, and finally check that $\mathbf{X}(3)^4 = \mathbf{X}(7)\mathbf{X}(3) = [1, 1, 1, 1] = \mathbf{X}(1)$.

If G is not cyclic then, since it is Abelian, it will be isomorphic to the direct product of a finite number of cyclic groups $G \cong C_1 \times C_2 \times \cdots \times C_m$. We can form m "generator" tuples for each of the generators, $a_1, \ldots, a_m$, of the smaller cyclic groups, using appropriate subgroups of the $\phi(k)$th roots of unity. The complete set of tuples is found by computing the Cartesian products of the generated sets of tuples.

Now we can arrange the nonzero values of χ in a square matrix A of size $\phi(k)$ with the tuples $\mathbf{X}$ described above forming the columns. The rows are then the nonzero values of the characters $\chi_1, \chi_2, \ldots, \chi_{\phi(k)}$. The elements of the first row and first column all equal one and so sum to $\phi(k)$, and we find that the sum of the elements of the other rows and columns is always zero.

This illustrates two *orthogonality* properties of Dirichlet characters: $\sum_{n \bmod k} \chi(n) = \phi(k)$ if $\chi = \chi_1$, or zero otherwise (across the rows); and $\sum_{\chi \bmod k} \chi(r) = \phi(k)$ if $r \equiv 1 \pmod{k}$, or zero otherwise (down the columns), where the subscript χ mod k means χ ranges over all the characters modulo k. These properties can be used as an extra check on any table of characters.