Divisibility by 13

Rule. *A number is divisible by 13 if the number formed by removing the last digit and adding four times that digit to the remaining number is also divisible by 13.*

Proof. Let $n = 10a + b$ where b is the last digit of n. We want to show that n is divisible by 13 if and only if $13 \mid a + 4b$.

Suppose $13 \mid a + 4b$ then

$$\begin{aligned} a + 4b &\equiv 0 \pmod{13} \\ a &\equiv -4b \pmod{13} \\ 10a &\equiv -40b \pmod{13}, \quad \text{multiplying each side by 10} \\ &\equiv -b \pmod{13}, \quad \text{since } 40 \equiv 1 \pmod{13} \\ 10a + b &\equiv 0 \pmod{13}. \end{aligned}$$

Thus $13 \mid a + 4b \implies 13 \mid 10a + b$, i.e. $13 \mid n$.

Conversely, suppose $13 \mid 10a + b$ then we have

$$\begin{aligned} 10a + b &\equiv 0 \pmod{13} \\ 10a &\equiv -b \pmod{13} \\ 10^{-1} \cdot 10a &\equiv 10^{-1} \cdot (-b) \pmod{13}, \quad \text{multiplying each side by } 10^{-1} \equiv 4 \pmod{13} \\ a &\equiv -4b \pmod{13} \\ a + 4b &\equiv 0 \pmod{13}. \end{aligned}$$

Thus $13 \mid 10a + b \implies 13 \mid a + 4b$.

Hence we have the required result, $13 \mid a + 4b \iff 13 \mid n$. ☐

This procedure is recursive and can be repeated until the result is small enough to spot that it is divisible by 13 (or not).

Examples

$13 \mid 52$ since $5 + 4 \times 2 = 13$ and $13 \mid 13$
$13 \mid 169$ since $16 + 4 \times 9 = 52$ and $13 \mid 52$
$13 \mid 182$ since $18 + 4 \times 2 = 26$ and $13 \mid 26$
$13 \mid 195$ since $19 + 4 \times 5 = 39$ and $13 \mid 39$