Divisibility by 7

Rule. *A number is divisible by 7 if the number formed by removing the last digit and subtracting two times that digit from the remaining number is also divisible by 7.*

Proof. Let $n = 10a + b$ where b is the last digit of n. We want to show that n is divisible by 7 if and only if $7 \mid a - 2b$.

Suppose $7 \mid a - 2b$ then

$$\begin{aligned}
a - 2b &\equiv 0 \pmod 7 \\
a &\equiv 2b \pmod 7 \\
10a &\equiv 20b \pmod 7, \quad \text{multiplying each side by 10} \\
&\equiv -b \pmod 7, \quad \text{since } 20 \equiv -1 \pmod 7 \\
10a + b &\equiv 0 \pmod 7.
\end{aligned}$$

Thus $7 \mid a - 2b \implies 7 \mid 10a + b$, i.e. $7 \mid n$.

Conversely, suppose $7 \mid 10a + b$ then we have

$$\begin{aligned}
10a + b &\equiv 0 \pmod 7 \\
10a &\equiv -b \pmod 7 \\
3a &\equiv -b \pmod 7, \quad \text{since } 10 \equiv 3 \pmod 7 \\
3^{-1} \cdot 3a &\equiv 3^{-1} \cdot (-b) \pmod 7, \quad \text{multiplying each side by } 3^{-1} \equiv 5 \pmod 7 \\
a &\equiv -5b \pmod 7 \\
&\equiv 2b \pmod 7, \quad \text{since } -5 \equiv 2 \pmod 7 \\
a - 2b &\equiv 0 \pmod 7.
\end{aligned}$$

Thus $7 \mid 10a + b \implies 7 \mid a - 2b$.

Hence we have the required result, $7 \mid a - 2b \iff 7 \mid n$. □

This procedure is recursive and can be repeated until the result is small enough to spot that it is divisible by 7 (or not).

Examples

$7 \mid 91$ since $9 - 2 \times 1 = 7$ and $7 \mid 7$
$7 \mid 119$ since $11 - 2 \times 9 = -7$ and $7 \mid -7$
$7 \mid 42$ since $4 - 2 \times 2 = 0$ and $7 \mid 0$